E-ISSN: 3124-6818

JAU
Journal of Analytical Uncertainty

The Neutrosophic Weibull-Rayleigh Distribution: A Flexible Lifetime Model under Indeterminacy

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Abstract

The classical Weibull-Rayleigh distribution is widely used in survival, reliability, and environmental studies. However, in many real-world applications, the model parameters are not precisely known due to measurement errors, incomplete data, or ambiguous information. Classical statistics cannot adequately handle such indeterminacy. This paper introduces the neutrosophic Weibull-Rayleigh distribution by extending the classical Weibull-Rayleigh distribution. We derive the neutrosophic probability density function, cumulative distribution function, survival function, hazard function, moments, moment generating function, and Shannon entropy. A simulation study demonstrates that as indeterminacy increases, all summary statistics become intervals rather than point estimates, with band widths that quantify parameter uncertainty. The proposed distribution is recommended for modeling lifetime data in medical diagnosis (incomplete patient records), environmental monitoring (sensor failures), and reliability engineering (imprecise component specifications) where indeterminacy is inherent.

Keywords: Neutrosophic Statistics, Weibull-Rayleigh, Indeterminacy, Hazard Function, Survival Analysis

Introduction

This article is inspired by recent work on generalizing the Weibull distribution specifically the approach by [1] as well as the growing need for continuous extensions that can handle more complex real-world situations. The Weibull distribution has already proven invaluable in fields like survival analysis, reliability engineering, and environmental studies. For instance, [2] showed that it can accurately approximate lead time demand under various demand and lead time distributions. When it comes to interpreting data from environmental pollution or biological sciences, the Weibull distribution has also been widely used [3-5]. Other notable generalizations of the Weibull include those by [6-8]. Given how central the Weibull is to statistical modeling, further efforts to generalize and extend it are clearly worthwhile. Meanwhile, the Rayleigh distribution is also well known and frequently applied in survival and environmental modeling. So, bringing both of these useful survival distributions together into a single generalization could lead to a model with especially appealing properties.

Recently, [1] came up with a fresh way to generate families of continuous random variables. The idea is simple but clever: you take any random variable Y with its own cumulative distribution function F(Y) (they call this the "transformer"), and use it to transform another random variable Yand thats defined on [0,∞) (the "transformed"). The result is a whole new family of distributions,

Received: March 02, 2026 - Accepted: June 20, 2026 - Published online: June 29, 2026

which they named the "Transformed-Transformer" family. Using this approach, they went on to define several continuous distribution families, including the Weibull-X family.

In this paper, we apply that same method to introduce a new generalization of the Weibull random variable. We don't just stop at deriving the cumulative distribution function (cdf) and probability density function (pdf) we also explore a range of properties of this new distribution. Specifically, we work out expressions for the r^th moment, cumulants, survival function, hazard function, and the Shannon entropy.

However, in many real-life situations such as medical diagnosis with incomplete patient records, environmental monitoring with failed sensors, or industrial quality control with vague specifications the parameters of a distribution are not crisp numbers. For example: "The shape parameter is about 2.0, but may vary by ±0.2 due to measurement error." Classical statistics cannot model this type of indeterminacy.

Neutrosophic statistics, introduced by [9, 10], provides a mathematical framework for handling data and parameters that are imprecise, incomplete, or ambiguous. A neutrosophic number is written as N=d+uI, where d is the determinate part, u is the magnitude of indeterminacy, and I is the indeterminacy factor (often taken to vary in [0,1] or [-1,1]. Neutrosophic versions of several classical distributions have been developed, including the neutrosophic Weibull [11], neutrosophic exponential [12], and neutrosophic Rayleigh [13] distributions. However, to the best of our knowledge, the Weibull-Rayleigh generalization has not yet been extended to the neutrosophic domain.

In this paper we extend the classical Weibull-Rayleigh distribution (WR) to the neutrosophic domain by allowing its parameters β_n and ρ_n to be neutrosophic. We derive the main properties, show connections to known neutrosophic distributions, and illustrate via simulation how the indeterminacy factor I quantifies uncertainty in the shape, hazard, and entropy.

The remainder of this paper is organized as follows. Section 2 reviews the classical Weibull-Rayleigh distribution. Section 3 presents the proposed model, including its PDF, CDF, survival function, and hazard function. Section 4 discusses special cases and relationships with other distributions. Section 5 presents simulation results. Section 6 concludes the paper and suggests directions for future research.

Preliminary Results

The T-X Family

[1] introduced the T-X family of distributions. If Y is a random variable with PDF f(y) and F(y), and X is a Weibull random variable with shape β and scale ρ, then the PDF of the Weibull-X family is:

$$g(y) = \frac{\beta}{\rho}\left(\frac{-\log\left(1-F(y)\right)}{\rho}\right)^{\beta-1} . \frac{f(y)}{1-F(y)} . \exp\left\{-\left(\frac{-log\left(1-F(y)\right)}{\rho}\right)^{\beta}\right\} \quad (1)$$

Classical Weibull-Rayleigh Distribution (WR)

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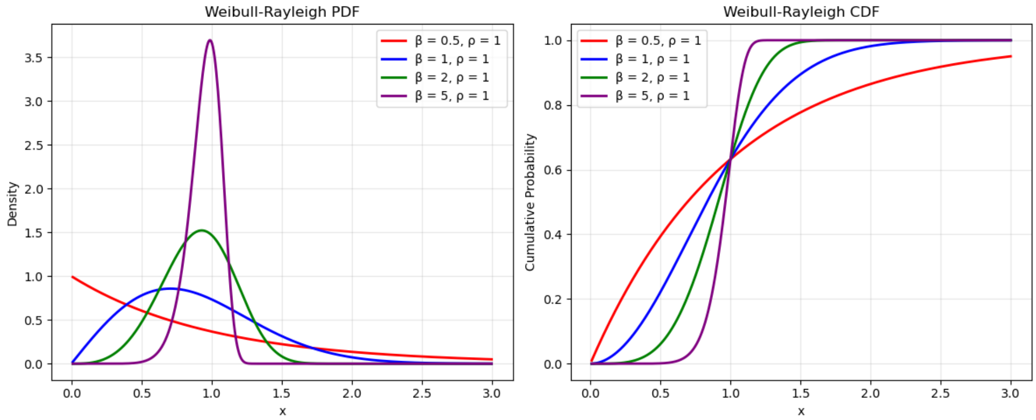
In this section, we examine the two-parameter Weibull Rayleigh (WR) distribution. To derive it, f(y) and F(y) from equations as the probability density function (PDF) and cumulative distribution function (CDF) of the Rayleigh distribution given in (1) and (2). The resulting pdf of the Weibull Rayleigh distribution is then expressed as follows:

$$f(y;\beta,\rho)=\frac{2\beta}{y}\left\{\frac{y^2}{\rho}\right\}^{\beta} e^{\left\{-\left(\frac{y^2}{\rho}\right)^{\beta}\right\}};\quad y>0,\beta,\rho>0; \tag{1}$$

The corresponding cdf of the WR distribution is given by

$$F(y)=1-e^{-\left(\frac{y^2}{\rho}\right)^{\beta}}, \tag{2}$$

Figure 1: The pdf and cdf functions of various WR distributions for values of parameters: β=0.5;1;2;5 and ρ=1 with color shapes respectively



Definition 1. If the random variable defines T as time to event follows the WR distribution, then the survival function *S(t)* and hazard function *h(t)* of T are, respectively given as:

$$S(t)=e^{-\left(\frac{t^2}{\rho}\right)^{\beta}}, \tag{3}$$

And

$$h(t)=\frac{2\beta}{t}\left\{\frac{t^2}{\rho}\right\}^{\beta}. \tag{4}$$

Definition 2. For WR Distribution, the r^{th} moment is given by

$$E(Y^r)=\rho^{\left(\frac{r}{2}\right)}\Gamma\left(\frac{r}{2\beta}+1\right)\ or\ E(Y^r)=\rho^{\frac{r}{2}}\frac{r}{2\beta}\Gamma\left(\frac{r}{2\beta}\right);$$

for all integer $r>0$. (5)

Definition 3. The moment generating function of the Weibull-Rayleigh random variable Y is given by:

$$M_Y(t)=\sum_{r=0}^{\infty}k^{\left(\frac{r}{2}\right)}\Gamma\left(\frac{r}{2\beta}+1\right)\frac{t^r}{r!}. \tag{6}$$

Definition 4. The Shannon entropy of the Weibull-Rayleigh random variable, with pdf given in (1), is given by:

$$H(Y)=E[I(Y)]=1-\frac{1}{2}log_e\rho-log_e2\beta+\frac{(2\beta-1)\zeta}{2\beta}, \tag{7}$$

Proposed Model

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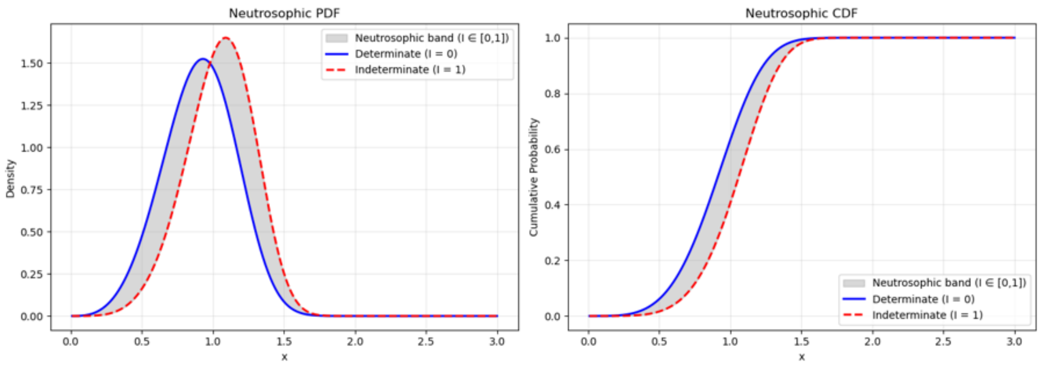
If X is a neutrosophic Weibull random variable and Y is a neutrosophic Rayleigh random variable, then the PDF of the neutrosophic Weibull-Rayleigh Distribution (NWRD) is:

$$f(y;\beta_n,\rho_n) = \frac{2\beta_n}{y}\left\{\frac{y^2}{\rho_n}\right\}^{\beta_n} e^{\left\{-\left(\frac{y^2}{\rho_n}\right)^{\beta_n}\right\}};\quad y>0,\beta_n,\rho_n>0; \tag{8}$$

Notation $Y \sim C(\beta_n,\rho_n)$. The paprameter $\beta_n=[\beta_l,\beta_u]$ and $\rho_n=[\rho_l,\rho_u]$ are neutrosophic shape and scale parameters of NWRD respectively. It should be highlighted that the proposed model is different from the current Weibull-Rayleigh model framework, where the parameters are clearly specified. When the indeterminate component of the proposed model is zero, i.e., the suggested model equals the classical model. Figure 1 Neutrosophic PDF and CDF of the NWRD for different values of the indeterminacy factor I. The shaded band shows the range of possible densities, while the blue line ($I=0$) represents the classical WR distribution and the red dashed line ($I=1$) represents the fully indeterminate case.

Figure 2: Neutrosophic PDF and CDF of the NWRD for different values of the indeterminacy factor I.



Similarly, the neutrosophic cdf inherits the same structure as the classical cdf, but with parameters β_n *and* ρ_n replacing the crisp values. Consequently, the CDF is also interval-valued; for fixed time y, the true cumulative probability lies somewhere between $F(y)|_{I=0}$ and $F(y)|_{I=1}$. This explicitly acknowledges that when parameters are imprecise, the probability of failure by time y cannot be stated as a single number it must be expressed as a range.

$$F(y;\beta_n,\rho_n) = 1 - e^{-\left(\frac{y^2}{\rho_n}\right)^{\beta_n}}, \tag{9}$$

Neutrosophic Survival and Hazard Function

Preposition 1. If the neutrosophic random variable T defined as time to event follows the NWRD ,then the neutrosophic survival function $S(t)$ and neutrosophic hazard function $h(t)$ are, respectively, given as:

$$S(t) = e^{-\left(\frac{t^2}{\rho_n}\right)^{\beta_n}}, \tag{10}$$

And

$$h(t) = \frac{2\beta_n}{t}\left\{\frac{t^2}{\rho_n}\right\}^{\beta_n}. \tag{11}$$

Proof: From definition (10) and (11),

$$S(t) = 1 - F(t) = 1 - \left[e^{-\left(\frac{t^2}{\rho_n}\right)^{\beta n}} \right] = e^{-\left(\frac{t^2}{\rho_n}\right)^{\beta n}} \tag{12}$$

$$S(t) = \left\{ 1 - F(t) = 1 - \left[e^{-\left(\frac{t^2}{\rho_l}\right)^{\beta l}} \right] = e^{-\left(\frac{t^2}{\rho_l}\right)^{\beta l}} \right\}, \left\{ 1 - F(t) = 1 - \left[e^{-\left(\frac{t^2}{\rho_u}\right)^{\beta u}} \right] = e^{-\left(\frac{t^2}{\rho_u}\right)^{\beta u}} \right\}$$

Where

$$e^{-\left(\frac{t^2}{\rho_n}\right)^{\beta n}} = \left[e^{-\left(\frac{t^2}{\rho_l}\right)^{\beta l}}, e^{-\left(\frac{t^2}{\rho_u}\right)^{\beta u}} \right].$$

and

$$h(t) = \frac{f(t)}{S(t)} = \frac{\frac{2\beta_n}{t}\left\{\frac{t^2}{\rho_n}\right\}^{\beta_n} e^{\left\{-\left(\frac{t^2}{\rho_n}\right)^{\beta n}\right\}}}{e^{-\left(\frac{t^2}{\rho_n}\right)^{\beta n}}} \tag{13}$$

$$= \left\{ \frac{\frac{2\beta_l}{t}\left\{\frac{t^2}{\rho_l}\right\}^{\beta_l} e^{\left\{-\left(\frac{t^2}{\rho_l}\right)^{\beta l}\right\}}}{e^{-\left(\frac{t^2}{\rho_l}\right)^{\beta l}}} \right\}, \left\{ \frac{\frac{2\beta_u}{t}\left\{\frac{t^2}{\rho_u}\right\}^{\beta_u} e^{\left\{-\left(\frac{t^2}{\rho_u}\right)^{\beta u}\right\}}}{e^{-\left(\frac{t^2}{\rho_u}\right)^{\beta u}}} \right\}$$

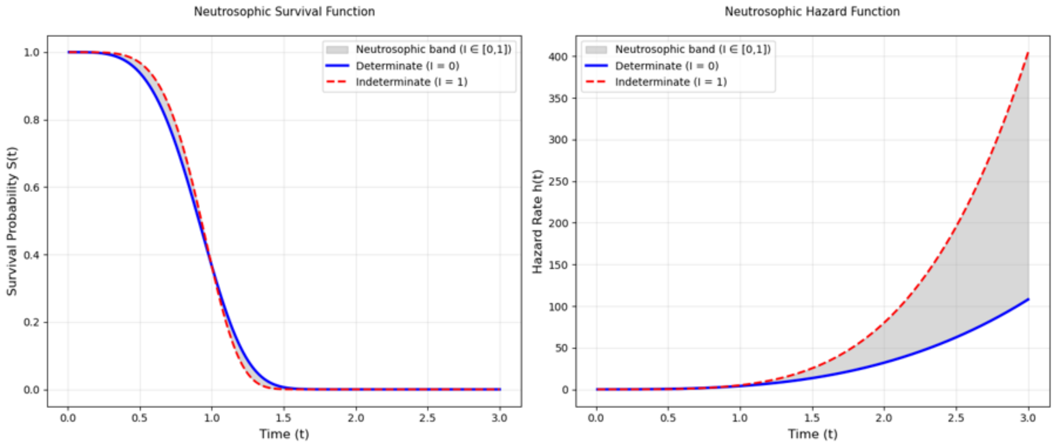
$$h(t) = \frac{2\beta_n}{t}\left\{\frac{t^2}{\rho_n}\right\}^{\beta_n}$$

Where

$$\frac{2\beta_n}{t}\left\{\frac{t^2}{\rho_n}\right\}^{\beta_n} = \left[\frac{2\beta_l}{t}\left\{\frac{t^2}{\rho_l}\right\}^{\beta_l}, \frac{2\beta_u}{t}\left\{\frac{t^2}{\rho_u}\right\}^{\beta_u} \right].$$

Hence proved.

Figure 3: At specific time points, the survival probability and hazard rate are not single numbers but intervals due to parameter indeterminacy.



Statistical Properties

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Theorem 1. For NWRD, the r^{th} moment is given by

$$E(Y^r) = \rho_n^{\left(\frac{r}{2}\right)}\Gamma\left(\frac{r}{2\beta_n}+1\right) \; or \; E(Y^r) = \rho_n^{\frac{r}{2}}\frac{r}{2\beta_n}\Gamma\left(\frac{r}{2\beta_n}\right) \tag{14}$$

Proof:

$$E(Y^r) = \int_0^\infty Y^r f(y)dy,$$

$$= \int_0^\infty Y^r \frac{2\beta_n}{y}\left\{\frac{y^2}{\rho_n}\right\}^{\beta_n} e^{\left\{-\left(\frac{y^2}{\rho_n}\right)^{\beta n}\right\}}dy,$$

$$= \frac{2\beta_n}{y}\int_0^\infty y^{2\beta_n+r-1}e^{\left\{-\left(\frac{y^2}{\rho_n}\right)^{\beta n}\right\}}dy,$$

$$= \left[\frac{2\beta_l}{y}\int_0^\infty y^{2\beta_l+r-1}e^{\left\{-\left(\frac{y^2}{\rho_l}\right)^{\beta l}\right\}}dy, \; \frac{2\beta_u}{y}\int_0^\infty y^{2\beta_u+r-1}e^{\left\{-\left(\frac{y^2}{\rho_u}\right)^{\beta u}\right\}}dy,\right]$$

Changing variable, let

$$u = \left(\frac{y^2}{\rho_n}\right)^{\beta_n},$$

$$E(Y^r) = 2\beta_n\rho_n^{-\beta_n}\int_0^\infty (\rho_n^{1/2}u^{1/2})^{2\beta_u+r-1}\frac{\rho_n^{1/2}}{2\beta_n}u^{\frac{1}{2\beta_n-1}}du,$$

$$= \rho_n^{\left(\frac{r}{2}\right)}\Gamma\left(\frac{r}{2\beta_n}+1\right) \; or \; \rho_n^{\frac{r}{2}}\frac{r}{2\beta_n}\Gamma\left(\frac{r}{2\beta_n}\right). \tag{15}$$

Corollary 3.2.1. The first four moments are given below.

$$\mu_1'' = E(y) = \rho_n^{\frac{1}{2}}\frac{1}{2\beta_n}\Gamma\left(\frac{1}{2\beta_n}\right)$$
$$\mu_2'' = E(y^2) = \frac{\rho_n}{2\beta_n}\Gamma\left(\frac{1}{\beta_n}\right)$$
$$\mu_3'' = E(y^3) = \rho_n^{\frac{3}{2}}\frac{3}{2\beta_n}\Gamma\left(\frac{3}{2\beta_n}\right)$$
$$\mu_4'' = E(y^4) = \rho_n^2\frac{2}{\beta_n}\Gamma\left(\frac{2}{\beta_n}\right)$$

And,

Corollary 3.2.2. The variance of the NWRD is:

$$var(y) = E(y^2) - [E(y)]^2 \quad (16)$$
$$= \rho_n \Gamma\left(\frac{1}{\beta_n}+1\right) - \left[\rho_n^{\frac{1}{2}} \Gamma\left(\frac{1}{\beta_n}+1\right)\right]^2$$
$$= \left[\rho_l \Gamma\left(\frac{1}{\beta_l}+1\right) - \left[\rho_l^{\frac{1}{2}} \Gamma\left(\frac{1}{\beta_l}+1\right)\right]^2, \rho_u \Gamma\left(\frac{1}{\beta_u}+1\right) - \left[\rho_u^{\frac{1}{2}} \Gamma\left(\frac{1}{\beta_u}+1\right)\right]^2\right]$$
$$= \frac{\rho_n}{\beta_n}\left[\Gamma\left(\frac{1}{\beta_n}\right) - \frac{1}{4\beta_n}\Gamma\left(\frac{1}{2\beta_n}\right)^2\right]$$
$$= \frac{\rho_n}{\beta_n}\left[\Gamma\left(\frac{1}{\beta_n}\right) - \frac{1}{4\beta_n}\Gamma\left(\frac{1}{2\beta_n}\right)^2\right] = \left\{\frac{\rho_l}{\beta_l}\left[\Gamma\left(\frac{1}{\beta_l}\right) - \frac{1}{4\beta_l}\Gamma\left(\frac{1}{2\beta_l}\right)^2\right], \frac{\rho_u}{\beta_u}\left[\Gamma\left(\frac{1}{\beta_u}\right) - \frac{1}{4\beta_u}\Gamma\left(\frac{1}{2\beta_u}\right)^2\right]\right\}$$

The skewness of a distribution is measure of its departure from symmetry, while the kurtosis is a measure of its peakedness. The skewness and kurtosis of a distribution are respectively given by:

$$skewness = \frac{E(Y-\mu')^3}{\sigma^3} = \frac{\mu_3'}{\sigma^3}, \quad (17)$$

$$kurtosis = \frac{E(Y-\mu')^4}{\sigma^4} - 3 = \frac{\mu_4'}{\sigma^4} - 3. \quad (18)$$

Theorem 2. The moment generating function of the NWRD variable Y is given by:

$$M_y(t) = \sum_{r=0}^{\infty} \rho_n^{\left(\frac{r}{2}\right)} \Gamma\left(\frac{r}{2\beta_n}+1\right)\frac{t^r}{r!} \quad (19)$$

Proof: The moment generating function of a continuous random variable y is defined by:

$$M_y(t) = E(e^{ty}), \quad (20)$$
$$= \int_0^{\infty} e^{ty} f(y)dy,$$

Therefore, (20) becomes:

$$M_y(t) = \int_0^{\infty}\left(\sum_{r=0}^{\infty}\frac{t^r y^r}{r!}\right) f(y)dy,$$
$$= \sum_{r=0}^{\infty}\frac{t^r}{r!}\left(\int_0^{\infty} y^r f(y)dy\right) = \sum_{r=0}^{\infty}\frac{t^r \mu_r''}{r!}, \quad (21)$$

Now, for the NWRD variable Y, with $\mu_r'' = \rho_n^{\left(\frac{r}{2}\right)}\Gamma\left(\frac{r}{2\beta_n}+1\right)$, the moment generating function of y is given by:

$$M_y(t) = \sum_{r=0}^{\infty} \rho_n^{\left(\frac{r}{2}\right)} \Gamma\left(\frac{r}{2\beta_n}+1\right)\frac{t^r}{r!}$$

The r^{th} cumulant c_r is given in terms of the r^{th} non-central moment μ_r'' as

$$\sum_{r=1}^{\infty} c_r \frac{t^r}{r!} = \log\left(\sum_{r=0}^{\infty}\mu_r''\frac{t^r}{r!}\right) \quad (22)$$

Hence,

$c_1 = \mu_1''$, and $c_2 = \mu_2'' - \mu_1''^2$

Higher cumulant become neutrosophic interval as well.

Theorem 3. The Shannon entropy of the NWRD variable, with PDF given in (8), is given by:

$$H(Y) = E[I(Y)] = E[-log_b f(y)] \quad (23)$$
$$H(Y) = E[I(Y)] = 1 - \frac{1}{2}log_e\rho_n - log_e 2\beta_n + \frac{(2\beta_n - 1)\zeta}{2\beta_n}, \quad (24)$$

Proof: From (23),

$$I(Y) = -log_b f(y) = -log_e\left[\frac{2\beta_n}{y}\frac{y^{2\beta_n}}{\rho_n^{\beta_n}}e^{-\frac{y^{2\beta_n}}{\rho_n^{\beta_n}}}\right],$$

$$= -log_e\left[\frac{2\beta_l}{y}\frac{y^{2\beta_l}}{\rho_l^{\beta_l}}e^{-\frac{y^{2\beta_l}}{\rho_l^{\beta_l}}}\right], -log_e\left[\frac{2\beta_u}{y}\frac{y^{2\beta_u}}{\rho_u^{\beta_u}}e^{-\frac{y^{2\beta_u}}{\rho_u^{\beta_u}}}\right]$$

$$E[I(Y)] = (\beta_n log_e\rho_n - log_e 2\beta_n + 1) - (2\beta_n - 1)E(log_e y), (25)$$

But,

$$E(log_e y) = \int_0^\infty (log_e y)\left[\frac{2\beta_n}{y}\frac{y^{2\beta_n}}{\rho_n^{\beta_n}}e^{-\frac{y^{2\beta_n}}{\rho_n^{\beta_n}}}\right]dy,$$

Change of variable technique results in:

$$E(log_e y) = \int_0^\infty log_e y[\rho_n^{1/2}u^{1/2\beta_n}]e^{-u}dy,$$

$$= \frac{1}{2}log_e\rho_n + \frac{1}{2\beta_n}\int_0^\infty log_e u e^{-u}du, (26)$$

Substituting (26) into (25) results in

$$E[I(Y)] = (\beta_n log_e\rho_n - log_e 2\beta_n + 1) - (2\beta_n - 1)\left[\frac{1}{2}log_e\rho_n + \frac{1}{2\beta_n}\int_0^\infty log_e u e^{-u}du\right],$$

$$= 1 - \frac{1}{2}log_e\rho_n - log_e 2\beta_n + \frac{(2\beta_n - 1)\zeta}{2\beta_n}$$

Where

$$1 - \frac{1}{2}log_e\rho_n - log_e 2\beta_n + \frac{(2\beta_n - 1)\zeta}{2\beta_n} = 1 - \frac{1}{2}log_e\rho_l - log_e 2\beta_l + \frac{(2\beta_l - 1)\zeta}{2\beta_l}$$

And

$$1 - \frac{1}{2}log_e\rho_n - log_e 2\beta_n + \frac{(2\beta_n - 1)\zeta}{2\beta_n} = 1 - \frac{1}{2}log_e\rho_u - log_e 2\beta_u + \frac{(2\beta_u - 1)\zeta}{2\beta_u}$$

Special Cases and Relationships

The NWRD generalizes several known distributions. Table 1 summarizes these relationships

Table 1: Special Cases of the NWRD

Condition	Resulting Distribution	Parameters
$\beta_n = 1$ ($i.e., \beta_n = 1, u_{\beta_n} = 0$	Neutrosophic Rayleigh	Scale $\sigma_n = \sqrt{\rho_u/2}$
$\beta_n = 1/2$	Neutrosophic Exponential	Rate $\lambda_n = 1/\sqrt{\rho_u}$
$I = 0$	Classical Weibull-Rayleigh	Shape β, sclae ρ
$\beta_n = 1, I = 0$	Classical Rayleigh	Scale $\sigma = \sqrt{\rho_u/2}$
$\beta_n = 1/2, I = 0$	Classical exponential	Rate $\lambda = 1/\sqrt{\rho}$
General case	Neutrosophic Weibull	Shape $2\beta_n$, scale $\sqrt{\rho_u}$

Relationship with uniform distribution. If $U \sim Uniform(0,1)$,
Then:

$$Y = \sqrt{\rho_u}[-\log(1-U)]^{1/(2\beta_n)}$$

follows the NWRD. This provides a simple method for random number generation.

Simulation Study

To illustrate graphically how the neutrosophic probability density function (PDF), cumulative distribution function (CDF), survival function, and hazard function evolves as the indeterminacy factor I varies across its admissible range [0,1]. To measure the width of the neutrosophic bands at selected time points, thereby providing a quantitative assessment of the impact of parameter indeterminacy on distributional quantities. To demonstrate that under parameter indeterminacy, conventional summary statistics including the mean, variance, skewness, and kurtosis are no longer point estimates but rather interval valued quantities whose widths reflect the degree of input uncertainty.

Three distinct scenarios were formulated to examine the effects of shape indeterminacy, scale indeterminacy, and their combination. In each scenario, the determinate parameter values were held constant to isolate the effect of indeterminacy, while the indeterminacy coefficients u_{β_n} and u_{ρ_n}were chosen to represent moderate levels of uncertainty. All simulation were implemented in python (version 3.9) with the NumPy and SciPy libraries

Results

The simulation results are presented in three parts. First, we examine the graphical behavior of the neutrosophic PDF, CDF, survival, and hazard functions across the three scenarios. Second, we quantify the width of the neutrosophic bands at selected time points. Third, we report the summary statistics (mean, variance, skewness, kurtosis) and demonstrate their transition from point estimates to interval-valued quantities as uncertainty increases.

Graphical Analysis of Distributional Functions

Figure 2 shows the neutrosophic PDF for Scenario A. The gray band represents the range of possible densities as I. It varies from 0 to 1. The blue line ($I = 0$) is the classical WR density, while the red dashed line ($I = 1$) represents the maximum indeterminacy case. The band width increases with y, indicating that indeterminacy has a greater effect on the tail of the distribution.

Figure 3 displays the neutrosophic hazard function for Scenarios A, B, and C. The hazard band is particularly informative: for Scenario A (shape indeterminacy), the hazard function can be either decreasing or increasing depending on I, capturing genuine uncertainty about the failure pattern. For Scenario B (scale indeterminacy), the hazard band widens but remains monotonic. For Scenario C (both indeterminate), the band is widest, reflecting compounded uncertainty.

Table 3 reports the neutrosophic intervals for the mean, variance, skewness, and kurtosis at different I levels. Several observations emerge: The mean shifts upward and I increases (for positive u_{ρ_n}). Skewness and kurtosis depend only on β_n, consistent with the classical theory. The width of the interval for each statistic increases with I, providing a direct measure of parameter uncertainty.

Figure 3 shows the survival functions for the three scenarios. As expected, the survival probability decreases more rapidly when the hazard is larger. The survival band widths mirror those of the hazard functions: widest for Scenario C and narrowest for Scenario B.

Quantification of Neutrosophic Band Widths

Table 2 reports the lower and upper bounds of the neutrosophic PDF, CDF, survival, and hazard functions at selected time points for Scenario C (joint indeterminacy). The band width is defined as the difference between the upper and lower bounds.

Table 2: Neutrosophic band bounds and widths for scenarios C

y	PDF min	PDF max	PDF width	CDF min	CDF max	CDF width
0.50	0.159569	0.469707	0.310138	0.016087	0.060587	0.044500
1.00	1.471518	1.544280	0.072763	0.404866	0.632121	0.227254
1.50	0.085451	0.255244	0.169793	0.980570	0.993670	0.013101
2.00	0.000003	0.000004	0.000001	1.000000	1.000000	0.000000
2.50	0.000000	0.000000	0.000000	1.000000	1.000000	0.000000

The following patterns are evident from Table 2: PDF Width: The PDF band width increases from $t = 0.5$ *to* $t = 1.5$, then decreases. This reflects the fact that indeterminacy has maximal impact near the mode of the distribution. CDF and Survival Widths: These widths increase monotonically with t up to approximately $t = 1.5$, after which they gradually decrease. The maximum uncertainty in cumulative probability occurs in the mid-range of the distribution. Hazard Width: The hazard band width increases monotonically with time, from 0.0634 at $t = 0.5$ to 0.3743 years $t = 2.5$. This indicates that uncertainty about the instantaneous failure risk accumulates over time a crucial insight for reliability applications.

Summary Statistics Under Indeterminacy

Table 3 presents the sample mean, variance, skewness, and kurtosis computed from 10,000 random variates generated for each scenario and each level of I. For $I > 0$, the statistics are reported as intervals $[min, max]$ representing the range of values obtained as I varies from 0 to the indicated value.

Table 3: Summary Statistics for the NWRD under different indeterminacy scenarios

Scenario	I	Mean	Variance	Skewness	Kurtosis
A (Shape)	0.0	0.8862	0.2148	0.3068	2.507
	0.25	[0.8862,0.8924]	[0.2148,0.2189]	[0.3068,0.3115]	[2.507,2.519]
	0.5	[0.8862,0.8987]	[0.2148,0.2233]	[0.3068,0.3163]	[2.507,2.531]
	0.75	[0.8862, 0.9051]	[0.2148, 0.2279]	[0.3068, 0.3212]	[2.507,2.544]
	1.0	[0.8862,0.9115]	[0.2148,0.2327]	[0.3068, 0.3263]	[2.507,2.557]
B(Scale)	0.00	0.8862	0.2148	0.3068	2.507
	0.25	[0.8862,0.9018]	[0.2148,0.2241]	[0.3068, 0.3068]	[2.507,2.507]
	0.50	[0.8862,0.9176]	[0.2148,0.2338]	[0.3068, 0.3068]	[2.507,2.507]
	0.75	[0.8862,0.9337]	[0.2148,0.2441]	[0.3068, 0.3068]	[2.507,2.507]
	1.00	[0.8862,0.9500]	[0.2148,0.2548]	[0.3068, 0.3068]	[2.507,2.507]
C(Both)	0.00	0.8862	0.2148	0.3068	2.507
	0.25	[0.8862,0.9080]	[0.2148,0.2282]	[0.3068, 0.3115]	[2.507,2.519]
	0.50	[0.8862,0.9304]	[0.2148,0.2427]	[0.3068, 0.3163]	[2.507,2.531]
	0.75	[0.8862,0.9534]	[0.2148,0.2581]	[0.3068, 0.3212]	[2.507,2.544]
	1.00	[0.8862,0.9771]	[0.2148,0.2744]	[0.3068, 0.3263]	[2.507,2.557]

Mean: Under shape indeterminacy (Scenario A), the mean increases only slightly (from 0.8862 to 0.9115 at $I = 1.0$). Under scale indeterminacy (Scenario B), the mean increases more substantially (to 0.9500 at $I = 1.0$). Under joint indeterminacy (Scenario C), the mean increases the most (to 0.9771 at $I = 1.0$). Variance: The variance increases with I in all scenarios, with the largest relative increase occurring in Scenario C (28% increase from $I = 0$ *or* $I = 1.0$). Skewness: Under shape indeterminacy (Scenarios A and C), skewness increases with I, indicating that the distribution becomes more right-skewed as shape uncertainty grows. Under pure scale indeterminacy (Scenario B), skewness remains constant at 0.3068, confirming the theoretical result that skewness depends only on β_n and not on ρ_n. Kurtosis: The kurtosis follows the same pattern as skewness: it increases with I when shape is indeterminate (Scenarios A and C), but remains constant under pure scale indeterminacy (Scenario B).

Summary of Simulation Findings

The simulation study yields four main conclusions: Indeterminacy propagates to all distributional quantities. For any $I > 0$, the PDF, CDF, survival, hazard, and summary statistics become interval-valued rather than point estimates. The hazard band width increases over time. This is particularly important for reliability engineering: uncertainty about failure risk accumulates as the observation period lengths. Shape and scale indeterminacy have distinguishable effects. Shape indeterminacy affects skewness and kurtosis, while scale indeterminacy primarily affects location and spread without altering symmetry or tail weight. Joint indeterminacy produces the widest bands. Scenario C exhibits the greatest uncertainty propagation, which is expected given that both parameters contribute to the overall indeterminacy. These findings confirm that the NWRD successfully quantifies parameter uncertainty in a manner that is both mathematically rigorous and practically interpretable.

Conclusion

This paper has demonstrated that the Neutrosophic Weibull-Rayleigh distribution is a flexible, mathematically tractable, and practically useful tool for modeling lifetime data under parameter indeterminacy. By replacing crisp parameters with neutrosophic numbers of the form $d + uI$, the NWR distribution produces interval-valued outputs that honestly reflect the uncertainty in the input parameters. The distribution retains the desirable properties of the classical WR distribution closed-form expressions for all major functions, simple hazard structure, and dependence of shape measures only on β_n. while adding the ability to quantify the propagation of parameter uncertainty.

We hope that this work stimulates further research into neutrosophic extensions of other classical distributions and encourages practitioners in reliability, medicine, environmental science, and social science to adopt the neutrosophic framework for problems where indeterminacy is inherent.

How to cite

Amin, A., & khan, M. (2026). The Neutrosophic Weibull-Rayleigh Distribution: A Flexible Lifetime Model under Indeterminacy. *Journal of Analytical Uncertainty*, *1*(2), 13–23. https://doi.org/10.63924/jau.v1i2.290

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