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Recursive Intuitionistic Fuzzy SuperHyperGraphs

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Abstract

Finite hypergraphs generalize ordinary graphs by allowing each hyperedge to connect an arbitrary nonempty subset of vertices, thereby providing a natural framework for genuinely multiway interactions. To represent hierarchical and multi-layer structures, SuperHyperGraphs further extend this framework via iterated powerset constructions, so that set-valued objects formed at one level may serve as vertices at higher levels. Independently, recursive hypergraphs enrich the edge structure by allowing a hyperedge to contain not only vertices but also lower-level hyperedges, yielding nested incidence relations under a prescribed recursion depth. In this paper, we unify these two directions and introduce Recursive Intuitionistic Fuzzy SuperHyper Graphs. The proposed model combines hierarchical supervertices, recursively defined superhyperedges, and intuitionistic fuzzy membership/non-membership grades in the sense of Atanassov, enabling the representation of higher-order systems that are simultaneously hierarchical, recursive, and uncertain. We formulate the structure on a well-founded recursive universe, establish the fundamental axioms (including vertex–edge consistency and covering conditions), and study basic structural properties. In particular, we discuss induced substructures, isomorphisms, and level-induced (depth-truncated) structures, and clarify how the model re duces to standard intuitionistic fuzzy hypergraph-type objects in special cases. The proposed framework provides a mathematically consistent foundation for modeling complex relational systems with nested inter actions and uncertainty across multiple levels of organization.

Keywords: Recursive SuperHyperGraph, HyperGraph, SuperHyperGraph, Recursive HyperGraph, Fuzzy SuperHyperGraph

Introduction

Graph-based models are a standard tool for representing relational data: vertices correspond to entities, and edges describe pairwise relationships [1]. However, many practical systems involve interactions among three or more entities at the same time, and such *higher-order* relations cannot be faithfully captured by ordinary binary edges alone. A finite *hypergraph* addresses this limitation by allowing each hyperedge to connect an arbitrary nonempty subset of vertices, making it suitable for modeling genuine multi-entity interactions [2,3].

In many applications, the relational structure is not only higher-order but also *hierarchical*. One natural way to represent this hierarchy is to iterate the powerset operation. This leads to finite *SuperHyperGraphs*, in which set-valued objects created at one stage can be treated as vertices at the next stage, producing nested collections of vertices and edges across multiple levels [4–6]. Accordingly, SuperHyperGraphs combine two expressive features in a single framework: higher-order connectivity (as in hypergraphs) and hierarchical aggregation (e.g., groups, groups

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of groups, and deeper nested units). Such a perspective is useful for layered systems, modular architectures, and multi-scale networked data. Related hierarchical representations have also appeared in complex-network modeling and applied scientific settings [7,8].

Throughout this paper, unless explicitly stated otherwise, the integer n in $\mathcal{P}^n(\cdot)$ and the level index of an n-SuperHyperGraph are assumed to be nonnegative. For quick reference, Table 1.1 compares ordinary graphs, hypergraphs, and n-SuperHyperGraphs from a structural viewpoint.

Recursive HyperGraphs

Another independent extension concerns *recursive* edge structure. In a *recursive hypergraph*, a hyperedge may contain not only vertices but also lower-level hyperedges, so incidence itself can be nested up to a fixed recursion depth [9,10]. Building on this idea, a *recursive superhypergraph* combines hierarchical supervertices

Table 1: Compact comparison of graphs, hypergraphs, and n-SuperHyperGraphs.

Aspect	*Graph*	*Hypergraph*	*n-SuperHyperGraph*
Objects	Vertices and edges.	Vertices and hyperedges.	n-supervertices and superhyperedges.
Type of connection	Exactly two vertices.	Any nonempty subset of vertices.	Any nonempty subset of supervertices (with nested vertex objects).
Formal representation	$G = (V, E)$, $E \subseteq \binom{V}{2}$.	$H = (V, E)$, $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$.	$\mathrm{SHG}^{(n)} = (V, E)$, $V \subseteq \mathcal{P}^n(V_0)$, $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$.
Primary focus	Pairwise relations.	Higher-order relations.	Higher-order and hierarchical (multi-level) relations.

with recursive superhyperedges; in particular, a superhyperedge may include both supervertices and lower level superhyperedges. This yields a unified object capable of expressing vertex hierarchy and edge recursion simultaneously [11].

For convenience, Table 1.2 summarizes the main differences between recursive hypergraphs and recursive superhypergraphs.

Table 2: Concise comparison of recursive hypergraphs and recursive superhypergraphs.

Aspect	**Recursive HyperGraphs**	**Recursive SuperHyperGraphs**
Underlying vertex domain	Ordinary vertices (atomic objects)	Hierarchical *supervertices* (typically elements of iterated powersets)
Edge contents	A hyperedge may contain vertices and *lower-level hyperedges* (edge recursion)	A superhyperedge may contain supervertices and *lower-level superhyperedges* (edge recursion layered on a vertex hierarchy)
Main source of nesting	Primarily through *edges containing edges*	Through *both* supervertices (vertex hierarchy) and superhyperedges (edge recursion)
Recursion parameters	Controlled by a prescribed recursion depth	Controlled by recursion depth, together with the superlevel n for the vertex hierarchy
Typical interpretation	Higher-order relations that may refer to other relations	Multi-scale entities together with relations that may refer to lower-level relations

From the above discussion, it is clear that frameworks such as uncertain graphs and SuperHyperGraphs are highly important for modeling complex relational systems. However, to the best of our knowledge, a unified framework that combines the ideas of *Intuitionistic Fuzzy Graphs* and *recursive SuperHyperGraphs* has not yet been formally established.

To bridge this gap, in this paper we unify these two directions and introduce *Recursive Intuitionistic Fuzzy SuperHyperGraphs*. The proposed model integrates hierarchical supervertices, recursively defined superhyperedges, and intuitionistic fuzzy membership/non-membership grades in the sense of Atanassov, thereby providing a mathematical framework for representing higher-order systems that are simultaneously hierarchical, recursive, and uncertain.

Preliminaries

This section fixes the notation and recalls the set-theoretic and combinatorial notions used throughout the paper. We first review hypergraphs and SuperHyperGraphs, then summarize the recursive extension of hypergraphs and SuperHyperGraphs, and finally recall the intuitionistic fuzzy hypergraph model.

SuperHyperGraphs

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We begin with the standard notion of a hypergraph and then introduce the iterated powerset construction that underlies SuperHyperGraphs.

Definition 2.1 (Hypergraph). [12,13] A hypergraph is an ordered pair H = (V, E) such that

• V isafinite set, whose elements are called vertices, and

• E isafinite family of nonempty subsets of V, called hyperedges.

Hence, unlike an ordinary graph, a hyperedge may join any number of vertices, not only two.

Definition 2.2 (n-fold iterated powerset). [14,15] Let X be a set Define

$$\mathcal{P}^1(X) := \mathcal{P}(X),$$

and for each $n \geq 1$, recursively set

$$\mathcal{P}^{n+1}(X) := \mathcal{P}\big(\mathcal{P}^n(X)\big).$$

When the empty set is excluded, we use the notation

$$\mathcal{P}^n_*(X) := \mathcal{P}^n(X) \setminus \{\emptyset\}.$$

Definition 2.3 (n-SuperHyperGraph). *(see [4,16]) Let V_0 be a finite nonempty base set, and define recursively*

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \in \mathbb{N} \cup \{0\}).$$

*For a fixed integer $n \geq 0$, an n-*SuperHyperGraph on V_0 *is a pair*

$$\mathrm{SHG}^{(n)} = (V, E)$$

such that

$$V \subseteq \mathcal{P}^n(V_0) \qquad \text{and} \qquad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

The elements of V are called n-supervertices*, and the elements of E are called* superhyperedges. *Equivalently, each superhyperedge is a nonempty subset of the supervertex set V.*

(n, k)-recursive SuperHyperGraphs

A *recursive hypergraph* extends the usual hypergraph model by allowing a hyperedge to contain not only vertices but also lower-level hyperedges. This permits nested incidence patterns, including self-referential behavior, under a bounded recursion depth [9,10].

Definition 2.4 (Depth-k powerset universe). *[9,10] Let S be a nonempty set and let $k \in \mathbb{N} \cup \{0\}$. Define a sequence of sets $(S_i)_{i\geq 0}$ by*

$$S_0 := S, \qquad S_i := \mathcal{P}\Big(\bigcup_{j=0}^{i-1} S_j\Big) \quad (i \geq 1).$$

The depth-k powerset universe *generated by S is defined as*

$$2_{S,k} := \mathcal{P}\Big(\bigcup_{i=0}^{k} S_i\Big).$$

Definition 2.5 (k-recursive hypergraph). *[9,10] Let V be a finite vertex set and let $k \in \mathbb{N} \cup \{0\}$. A* k-recursive hypergraph *is a pair*

$$H = (V,E)$$

for which

$$E \subseteq 2_{V,k} \setminus \{\emptyset\},$$

where $2_{V,k}$ *is the depth-*k *powerset universe from Definition 2.4 with* $S = V$.
In the special case $k = 0$, *we have* $2_{V,0} = \mathcal{P}(V)$, *so the condition becomes*

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\},$$

and H *is exactly an ordinary hypergraph.*

An (n,k)-recursive SuperHyperGraph combines two ingredients: (i) level-n supervertices obtained from iterated powersets, and (ii) depth-k recursive superhyperedges whose elements may include both supervertices and lower-level recursive objects. In this paper, we consider the well-founded setting (i.e., no membership cycles).

Definition 2.6 ((n,k)-recursive SuperHyperGraph). *Let* V_0 *be a base (ground) set, and let* $n,k \in \mathbb{N} \cup \{0\}$.

(Iterated powersets). Define

$$\mathcal{P}^0(V_0) = V_0, \qquad \mathcal{P}^{n+1}(V_0) = \mathcal{P}\big(\mathcal{P}^n(V_0)\big) \quad (n \geq 0).$$

A (n,k)-recursive SuperHyperGraph *is a pair*

$$\mathrm{RSHG}^{(n,k)} = (V, E)$$

satisfying:

(i) *(Hierarchical supervertex set)*

$$V \subseteq \mathcal{P}^n(V_0).$$

(ii) *(Recursive superhyperedge family)*

$$E \subseteq 2_{V,k} \setminus \{\emptyset\},$$

where $2_{V,k}$ *is the depth-*k *powerset universe generated from* $S = V$ *as in Definition 2.4.*

Example 2.1 (A concrete (1,1)-recursive SuperHyperGraph). *We give a small example of an* (n,k)*-recursive SuperHyperGraph with*

$$n = 1, \quad k = 1.$$

Step 1: Base set and supervertex level. Let the ground set be

$$V_0 = \{a,b,c\}.$$

Since $n = 1$, *supervertices must be elements of* $\mathcal{P}(V_0)$. *Define*

$$v_1 := \{a\}, \qquad v_2 := \{b\}, \qquad v_3 := \{a,b\},$$

and set

$$V := \{v_1,v_2,v_3\} \subseteq \mathcal{P}(V_0).$$

Hence V *is a valid set of* 1*-supervertices.*
*Step 2: Depth-*1 *recursive universe over* V. *By Definition 2.4, the recursive layers are*

$$S_0 = V, \qquad S_1 = \mathcal{P}(V).$$

Therefore,

$$2_{V,1} = \mathcal{P}\big(S_0 \cup S_1\big) = \mathcal{P}\big(V \cup \mathcal{P}(V)\big).$$

So each recursive superhyperedge is allowed to be a nonempty subset of

$$V \cup \mathcal{P}(V),$$

meaning that an edge may contain both supervertices (elements of V) and recursive objects (elements of $\mathcal{P}(V)$).

Step 3: Choose recursive objects. Define two elements of $\mathrm{P}(V)$*:*

$$r_1 := \{v_1, v_3\}, \qquad r_2 := \{v_2\}.$$

These are not supervertices (they are subsets of V), but they belong to $S_1 = \mathrm{P}(V)$ *and can therefore appear inside recursive superhyperedges.*

Step 4: Define the recursive superhyperedges. Let

$$e_1 := \{v_1, v_3, r_1\}, \quad e_2 := \{v_2, r_1, r_2\}.$$

Since each of v_1, v_2, v_3, r_1, r_2 belongs to $V \cup \mathrm{P}(V)$, we have

$$e_1, e_2 \subseteq V \cup \mathrm{P}(V),$$

and thus

$$e_1, e_2 \in \mathcal{P}\big(V \cup \mathcal{P}(V)\big) = 2_{V,1}.$$

Also, $e_1 \neq \emptyset$ and $e_2 \neq \emptyset$.

Now define

$$E := \{e_1, e_2\}.$$

Then

$$E \subseteq 2_{V,1} \setminus \{\emptyset\}.$$

Conclusion. The pair

$$\mathrm{RSHG}^{(1,1)} = (V, E)$$

is an (1,1)*-recursive SuperHyperGraph. Indeed:*

(i) $V \subseteq \mathrm{P}^1(V_0) = \mathrm{P}(V_0)$, *and*

(ii) $E \subseteq 2_{V,1} \setminus \{\emptyset\}$.

This example illustrates the key feature of recursion: the edge e_1 (and also e_2) contains not only supervertices (v_1, v_3 or v_2), but also a lower-level recursive object (r_1 or r_2), thereby combining hierarchical vertices and recursive edge contents in a single structure.

Intuitionistic Fuzzy Hypergraphs

An intuitionistic fuzzy set assigns to each element both a membership degree and a non-membership degree, subject to Atanassov's condition

$$0 \le \mu(x) + \nu(x) \le 1$$

[17]. This framework naturally extends to graph-type structures, including hypergraphs, by assigning such degrees to vertices and edge incidences [18–20].

Definition 2.7 (Intuitionistic Fuzzy Hypergraph). *(cf. [?,20]) Let V be a finite nonempty set. An* intuitionistic fuzzy hyperedge on V is an ordered pair

$$E = (\mu E, \nu E),$$

where $\mu E, \nu E : V \to [0,1]$ *satisfy*

$$0 \leq \mu E(v) + \nu E(v) \leq 1, \ \forall v \in V.$$

The support *of E is*

$$\operatorname{supp}(E) = \{ v \in V : \ \mu_E(v) > 0 \text{ or } \nu_E(v) < 1 \}.$$

An intuitionistic fuzzy hypergraph *is a pair*

$$H = (V, \mathrm{E}),$$

where $\mathrm{E} = \{E_1, \ldots, E_m\}$ *is a finite family of intuitionistic fuzzy hyperedges on V such that*

$$\bigcup_{j=1}^{m} \operatorname{supp}(E_j) = V.$$

The order *of* H *is* $|V|$, *and the number of hyperedges is* $|\mathcal{E}|$.

Results: Recursive Intuitionistic Fuzzy superhypergraph

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In this section, we introduce an intuitionistic fuzzy extension of (n,k)-recursive SuperHyperGraphs. The construction combines: (i) hierarchical supervertices (via iterated powersets), and (ii) recursive superhyperedge contents (via bounded recursive set formation), together with Atanassov-type membership/non-membership degrees.

Definition 3.1 (Recursive universe over a supervertex set). *Let V be a finite nonempty set (in our application,* $V \subseteq \mathrm{P}^n(V_0)$*), and let* $k \in \mathrm{N} \cup \{0\}$. *Define recursively*

$V \subseteq \mathcal{P}^n(V_0)$*), and let* $k \in \mathbb{N} \cup \{0\}$. *Define recursively*

$$S_0 := V, \qquad S_i := \mathcal{P}\Big(\bigcup_{t=0}^{i-1} S_t\Big) \quad (i \geq 1).$$

The recursive universe of depth k over V *is*

$$U_{V,k} := \bigcup_{i=0}^{k} S_i.$$

Equivalently, the family of all possible (crisp) recursive superhyperedge contents of depth at most k *is*

$$2_{V,k} := \mathcal{P}(U_{V,k}).$$

Definition 3.2 (Intuitionistic fuzzy recursive superhyperedge). Let V be a finite nonempty set and let $k \in \mathrm{N} \cup \{0\}$. Anintuitionistic fuzzy recursive superhyperedge of depth at most k on V is an ordered pair

$$\mathcal{E} = (\mu_{\mathcal{E}}, \nu_{\mathcal{E}}),$$

where

$$\mu_{\mathcal{E}}, \nu_{\mathcal{E}} : U_{V,k} \to [0,1]$$

satisfy Atanassov's constraint

$$0 \le \mu_{\mathcal{E}}(x) + \nu_{\mathcal{E}}(x) \le 1, \qquad \forall x \in U_{V,k}.$$

Its support *is defined by*

$$\mathrm{supp}(\mathcal{E}) := \{\, x \in U_{V,k} : \ \mu_{\mathcal{E}}(x) > 0 \text{ or } \nu_{\mathcal{E}}(x) < 1 \,\}.$$

We require $\mathrm{supp}(\mathcal{E}) \neq \emptyset$. *Since* $\mathrm{supp}(\mathcal{E}) \subseteq U_{V,k}$, *it follows automatically that*

$$\mathrm{supp}(\mathcal{E}) \in 2_{V,k} \setminus \{\emptyset\}.$$

Definition 3.3 (Recursive Intuitionistic Fuzzy (n,k)-SuperHyperGraph). Let V0 be a finite nonempty base set, and let n,k ∈ N ∪ {0}. A Recursive Intuitionistic Fuzzy (n,k)-SuperHyperGraph (abbreviated RIF-(n,k) SHG) on V0 is a tuple

$$\mathcal{H} = \big(V, \mu_V, \nu_V, \mathfrak{E}\big),$$

satisfying the following conditions:

(i) ***Hierarchical supervertex set:***

$$V \subseteq \mathcal{P}^n(V_0), \qquad V \neq \emptyset.$$

Elements of V *are called* n-supervertices.

(ii) ***Intuitionistic fuzzy supervertex grades:***

$$\mu_V, \nu_V : V \to [0,1]$$

satisfy

$$0 \le \mu_V(v) + \nu_V(v) \le 1, \qquad \forall v \in V.$$

(iii) ***Recursive intuitionistic fuzzy superhyperedge family:***

$$\mathfrak{E} = \{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$$

is a finite family of intuitionistic fuzzy recursive superhyperedges of depth at most k *on* V *in the sense of Definition 3.2; equivalently, for each* j,

$$\mathcal{E}_j = (\mu_j, \nu_j), \qquad \mu_j, \nu_j : U_{V,k} \to [0,1],$$

with

$$0 \le \mu_j(x) + \nu_j(x) \le 1 \quad (\forall x \in U_{V,k}),$$

and $\mathrm{supp}(\mathcal{E}_j) \neq \emptyset$.

(iv) ***Vertex–edge consistency on level-0 recursive objects (the supervertices):*** *for every* $j \in \{1, \ldots, m\}$ *and every* $v \in V (= S_0)$,

$$\mu_j(v) \le \mu_V(v), \qquad \nu_j(v) \ge \nu_V(v).$$

That is, the degree of membership of a supervertex in a recursive superhyperedge cannot exceed its global supervertex membership degree, and its non-membership degree cannot be smaller than the global supervertex non-membership degree.

(v) ***Covering condition on supervertices:***

$$V \subseteq \bigcup_{j=1}^{m} \mathrm{supp}(\mathcal{E}_j).$$

Hence every supervertex is involved in at least one recursive intuitionistic fuzzy superhyperedge.

Remark 3.1 (Hesitation degree). *For a supervertex $v \in V$, the* hesitation degree *is*

$$\pi_V(v) := 1 - \mu_V(v) - \nu_V(v) \in [0, 1].$$

Likewise, for each recursive object $x \in U_{V,k}$ and each recursive superhyperedge $\mathcal{E}_j$,

$$\pi_j(x) := 1 - \mu_j(x) - \nu_j(x) \in [0, 1].$$

Remark 3.2 (Why the recursion is well-founded). *The recursive universe $U_{V,k} = \bigcup_{i=0}^{k} S_i$ is built layer by layer. If $x \in S_i$ with $i \geq 1$, then by construction*

$$x \subseteq \bigcup_{t=0}^{i-1} S_t.$$

Hence every membership step $y \in x$ moves strictly to a lower layer. Therefore, no $\in$-cycle can occur inside $U_{V,k}$, and recursive incidence is automatically well-founded.

Example 3.1 (A concrete RIF-(1,1)-SHG). We construct a small Recursive Intuitionistic Fuzzy (1,1)-SuperHyperGraph to illustrate Definition 3.3.

Step 1: Base set and supervertices. *Let*

$$V_0 = \{a, b, c\}.$$

Since $n = 1$, the supervertices must lie in $\mathcal{P}(V_0)$. Define

$$v_1 := \{a\}, \qquad v_2 := \{b\}, \qquad v_3 := \{a, b\},$$

and set

$$V := \{v_1, v_2, v_3\} \subseteq \mathcal{P}(V_0).$$

Thus V is a nonempty set of 1-supervertices.

Step 2: Intuitionistic fuzzy grades on supervertices. *Define $\mu_V, \nu_V : V \to [0, 1]$ by*

$$(\mu_V(v_1), \nu_V(v_1)) = (0.90, 0.05), \qquad (\mu_V(v_2), \nu_V(v_2)) = (0.80, 0.10), \qquad (\mu_V(v_3), \nu_V(v_3)) = (0.85, 0.05).$$

These satisfy Atanassov's condition:

$$0.90 + 0.05 = 0.95 \leq 1, \qquad 0.80 + 0.10 = 0.90 \leq 1, \qquad 0.85 + 0.05 = 0.90 \leq 1.$$

Step 3: Recursive universe of depth $k = 1$. *For $k = 1$, the recursive layers are*

$$S_0 = V, \qquad S_1 = \mathcal{P}(V).$$

Hence

$$U_{V,1} = S_0 \cup S_1 = V \cup \mathcal{P}(V).$$

To explicitly display the recursive objects (elements of S_1), define

$$r_1 := \{v_1, v_3\} \in \mathcal{P}(V), \qquad r_2 := \{v_2\} \in \mathcal{P}(V).$$

These are recursive-level objects *(they are sets of supervertices), so they can be assigned intuitionistic fuzzy grades inside recursive superhyperedges.*

Step 4: Recursive intuitionistic fuzzy superhyperedges. *Let*

$$\mathfrak{E} = \{\mathcal{E}_1, \mathcal{E}_2\}.$$

Each $\mathcal{E}_j = (\mu_j, \nu_j)$ is an intuitionistic fuzzy set on $U_{V,1}$. We define the nontrivial values as follows and assign the default value

$$(\mu_j(x), \nu_j(x)) = (0, 1) \qquad \text{for all other } x \in U_{V,1}.$$

For $\mathcal{E}_1$, set

$$(\mu_1(v_1), \nu_1(v_1)) = (0.70, 0.20), \qquad (\mu_1(v_3), \nu_1(v_3)) = (0.60, 0.25), \qquad (\mu_1(r_1), \nu_1(r_1)) = (0.80, 0.10).$$

Step 5: Supports of the recursive superhyperedges. *By definition,*

$$\mathrm{supp}(\mathcal{E}_j) = \{x \in U_{V,1} : \mu_j(x) > 0 \text{ or } \nu_j(x) < 1\}.$$

Therefore,

$$\mathrm{supp}(\mathcal{E}_1) = \{v_1, v_3, r_1\}, \qquad \mathrm{supp}(\mathcal{E}_2) = \{v_2, v_3, r_2\}.$$

Both supports are nonempty, and each support contains at least one recursive object (r_1 or r_2), so the example is genuinely recursive.

Verification of the axioms in Definition 3.3.

(i) Hierarchical supervertex set:

$$V = \{v_1, v_2, v_3\} \subseteq \mathcal{P}(V_0), \qquad V \neq \emptyset.$$

(ii) Intuitionistic fuzzy supervertex grades: *μ_V, ν_V are defined on V and satisfy*

$$0 \leq \mu_V(v) + \nu_V(v) \leq 1 \qquad (\forall v \in V).$$

(iii) Recursive intuitionistic fuzzy superhyperedge family: *Each $\mathcal{E}_j = (\mu_j, \nu_j)$ is defined on*

$$U_{V,1} = V \cup \mathcal{P}(V),$$

and satisfies

$$0 \leq \mu_j(x) + \nu_j(x) \leq 1 \qquad (\forall x \in U_{V,1}),$$

with $\mathrm{supp}(\mathcal{E}_j) \neq \emptyset$.

(iv) Vertex–edge consistency on supervertices: *for each $v \in V$ and each edge where v appears with nontrivial grade, we have*

$$\mu_j(v) \leq \mu_V(v), \qquad \nu_j(v) \geq \nu_V(v).$$

Indeed:

$$\begin{aligned}
\mu_1(v_1) &= 0.70 \leq 0.90 = \mu_V(v_1), & \nu_1(v_1) &= 0.20 \geq 0.05 = \nu_V(v_1),\\
\mu_1(v_3) &= 0.60 \leq 0.85 = \mu_V(v_3), & \nu_1(v_3) &= 0.25 \geq 0.05 = \nu_V(v_3),\\
\mu_2(v_2) &= 0.65 \leq 0.80 = \mu_V(v_2), & \nu_2(v_2) &= 0.25 \geq 0.10 = \nu_V(v_2),\\
\mu_2(v_3) &= 0.50 \leq 0.85 = \mu_V(v_3), & \nu_2(v_3) &= 0.30 \geq 0.05 = \nu_V(v_3).
\end{aligned}$$

(v) Covering condition on supervertices:

$$\mathrm{supp}(\mathcal{E}_1) \cup \mathrm{supp}(\mathcal{E}_2) = \{v_1, v_2, v_3, r_1, r_2\},$$

hence

$$V = \{v_1, v_2, v_3\} \subseteq \mathrm{supp}(\mathcal{E}_1) \cup \mathrm{supp}(\mathcal{E}_2).$$

So every supervertex is covered by at least one recursive intuitionistic fuzzy superhyperedge.

Therefore,

$$\mathcal{H} = \big(V, \mu_V, \nu_V, \mathfrak{E}\big)$$

is a valid Recursive Intuitionistic Fuzzy $(1,1)$-SuperHyperGraph. *This example also makes the two-level nature explicit: the objects v_1, v_2, v_3 are supervertices (level S_0), while $r_1, r_2 \in \mathcal{P}(V)$ are recursive objects (level S_1) that can participate in superhyperedges.*

Proposition 3.1 (Reduction to standard cases)**.** *Let $\mathcal{H}$ be a RIF-(n,k)-SHG as in Definition 3.3. Then:*

(a) *If $k = 0$, then $U_{V,0} = V$, and $\mathcal{H}$ reduces to an intuitionistic fuzzy n-SuperHyperGraph (with intuitionistic fuzzy superhyperedges defined directly on V).*

(b) *If $n = 0$, then $V \subseteq V_0$, so $\mathcal{H}$ becomes a recursive intuitionistic fuzzy hypergraph on an ordinary vertex set (with recursion depth bounded by k).*

(c) *If $n = 0$ and $k = 0$, then $\mathcal{H}$ reduces to an ordinary intuitionistic fuzzy hypergraph (on a crisp vertex set $V \subseteq V_0$).*

Proof. For $k = 0$, one has $S_0 = V$ and $U_{V,0} = V$, so each recursive intuitionistic fuzzy superhyperedge is simply an intuitionistic fuzzy set on V. For $n = 0$, supervertices are ordinary elements of the base set V_0 (up to the chosen subset V), while recursion in the edge contents is still allowed up to depth k. The case $(n,k) = (0,0)$ combines both reductions. □

Proposition 3.2 (Sub-RIF-SHG (induced recursive intuitionistic fuzzy sub-superhypergraph))**.** *Let*

$$\mathcal{H} = (V, \mu_V, \nu_V, \mathfrak{E})$$

be a RIF-(n,k)-SHG on a finite base set V_0, and let W be a nonempty subset of V. Define

$$\mu_W := \mu_V|_W, \qquad \nu_W := \nu_V|_W.$$

Let $U_{W,k}$ be the recursive universe of depth k over W as in Definition 3.1.

For each $\mathcal{E}_j = (\mu_j, \nu_j) \in \mathfrak{E}$, define its restriction to $U_{W,k}$ by

$$\mathcal{E}_j|_W := \big(\mu_j|_{U_{W,k}},\, \nu_j|_{U_{W,k}}\big).$$

Set

$$\mathfrak{E}[W] := \{\mathcal{E}_j|_W \ : \ \mathcal{E}_j \in \mathfrak{E},\ \mathrm{supp}(\mathcal{E}_j) \cap U_{W,k} \neq \emptyset\}.$$

If, in addition,

$$W \subseteq \bigcup_{\mathcal{F} \in \mathfrak{E}[W]} \mathrm{supp}(\mathcal{F}),$$

then

$$\mathcal{H}[W] := \big(W, \mu_W, \nu_W, \mathfrak{E}[W]\big)$$

is a RIF-(n,k)-SHG on the same base set V_0. We call $\mathcal{H}[W]$ the (induced) recursive intuitionistic fuzzy sub-superhypergraph *of $\mathcal{H}$ on W.*

Proof. Since $W \subseteq V \subseteq \mathcal{P}^n(V_0)$ and $W \neq \emptyset$, the hierarchical supervertex condition holds. Because μ_W, ν_W are restrictions of μ_V, ν_V, Atanassov's constraint

$$0 \le \mu_W(w) + \nu_W(w) \le 1 \qquad (\forall\, w \in W)$$

follows immediately.

For each $\mathcal{E}_j = (\mu_j, \nu_j) \in \mathfrak{E}$, the restricted maps $\mu_j|_{U_{W,k}}, \nu_j|_{U_{W,k}} : U_{W,k} \to [0,1]$ still satisfy Atanassov's constraint on $U_{W,k}$. By construction, only those restrictions with nonempty support are retained in $\mathfrak{E}[W]$.

It remains to check the vertex–edge consistency on W. Let $w \in W$ and $\mathcal{F} = \mathcal{E}_j|_W \in \mathfrak{E}[W]$. Because $w \in W \subseteq V = S_0 \subseteq U_{W,k}$,

$$\mu_{\mathcal{F}}(w) = \mu_j(w) \le \mu_V(w) = \mu_W(w), \qquad \nu_{\mathcal{F}}(w) = \nu_j(w) \ge \nu_V(w) = \nu_W(w),$$

where we used Definition 3.3(iv).

Finally, the assumed covering condition

$$W \subseteq \bigcup_{\mathcal{F} \in \mathfrak{E}[W]} \mathrm{supp}(\mathcal{F})$$

gives Definition 3.3(v) for $\mathcal{H}[W]$. Hence $\mathcal{H}[W]$ is a RIF-(n,k)-SHG. □

Proposition 3.3 (Isomorphism of RIF-(n,k)-SHGs)**.** *Let*

$$\mathcal{H} = (V, \mu_V, \nu_V, \mathfrak{E}), \qquad \mathcal{H}' = (V', \mu_{V'}, \nu_{V'}, \mathfrak{E}')$$

be two RIF-(n,k)-SHGs (possibly on different finite base sets). Write

$$U_{V,k} = \bigcup_{i=0}^{k} S_i, \qquad U_{V',k} = \bigcup_{i=0}^{k} S'_i$$

for the corresponding recursive universes.

Assume $f : V \to V'$ is a bijection. Then f extends uniquely to a bijection

$$\widehat{f} : U_{V,k} \to U_{V',k}$$

defined recursively by

$$\widehat{f}(x) = f(x) \quad (x \in V = S_0), \qquad \widehat{f}(x) = \{\widehat{f}(y) : y \in x\} \quad (x \in S_i,\ i \ge 1).$$

Suppose furthermore that:

(i) ***Vertex-grade preservation:***

$$\mu_{V'}(f(v)) = \mu_V(v), \qquad \nu_{V'}(f(v)) = \nu_V(v), \qquad \forall v \in V.$$

(ii) ***Edge-family preservation:*** *there exists a bijection*

$$\Phi : \mathfrak{E} \to \mathfrak{E}'$$

such that for every $\mathcal{E} = (\mu_{\mathcal{E}}, \nu_{\mathcal{E}}) \in \mathfrak{E}$ and every $x \in U_{V,k}$, if $\Phi(\mathcal{E}) = (\mu'_{\Phi(\mathcal{E})}, \nu'_{\Phi(\mathcal{E})})$, then

$$\mu'_{\Phi(\mathcal{E})}(\widehat{f}(x)) = \mu_{\mathcal{E}}(x), \qquad \nu'_{\Phi(\mathcal{E})}(\widehat{f}(x)) = \nu_{\mathcal{E}}(x).$$

Then (f, Φ) is called an isomorphism *from $\mathcal{H}$ to $\mathcal{H}'$, written*

$$\mathcal{H} \cong \mathcal{H}'.$$

Moreover, this notion of isomorphism is an equivalence relation on the class of RIF-(n,k)-SHGs.

Proof. The recursive definition of $\widehat{f}$ is well-defined because each element of S_i is a set of elements from $\bigcup_{t=0}^{i-1} S_t$, so the extension proceeds layer by layer. Bijectivity of $\widehat{f}$ follows by induction on the layer index i from bijectivity of f.

Under condition (ii), for every $\mathcal{E} \in \mathfrak{E}$ one has

$$x \in \mathrm{supp}(\mathcal{E}) \iff \big(\mu_{\mathcal{E}}(x) > 0 \text{ or } \nu_{\mathcal{E}}(x) < 1\big) \iff \big(\mu'_{\Phi(\mathcal{E})}(\widehat{f}(x)) > 0 \text{ or } \nu'_{\Phi(\mathcal{E})}(\widehat{f}(x)) < 1\big) \iff \widehat{f}(x) \in \mathrm{supp}(\Phi(\mathcal{E})),$$

hence

$$\widehat{f}\big(\mathrm{supp}(\mathcal{E})\big) = \mathrm{supp}(\Phi(\mathcal{E})).$$

Thus the full recursive incidence pattern and all intuitionistic fuzzy grades are preserved.

Finally, reflexivity is obtained by $f = \mathrm{id}_V$ and $\Phi = \mathrm{id}_{\mathfrak{E}}$. Symmetry follows from inverting the bijections (f, Φ). Transitivity follows by composition of isomorphisms and the uniqueness/compatibility of the recursive extensions. Therefore, $\cong$ is an equivalence relation. □

Proposition 3.4 (Level-induced structure (recursive-depth truncation))**.** *Let*

$$\mathcal{H} = (V, \mu_V, \nu_V, \mathfrak{E})$$

be a RIF-(n,k)-SHG, and let

$$U_{V,k} = \bigcup_{i=0}^{k} S_i$$

be its recursive universe. Fix $\ell \in \{0, 1, \ldots, k\}$ and define the truncated recursive universe

$$U_{V,\ell} := \bigcup_{i=0}^{\ell} S_i.$$

For each $\mathcal{E}_j = (\mu_j, \nu_j) \in \mathfrak{E}$, define the ℓ-level truncation

$$\mathcal{E}_j^{\langle \ell \rangle} := \big(\mu_j|_{U_{V,\ell}}, \, \nu_j|_{U_{V,\ell}}\big).$$

Let

$$\mathfrak{E}^{\langle \ell \rangle} := \left\{ \mathcal{E}_j^{\langle \ell \rangle} \;:\; \mathcal{E}_j \in \mathfrak{E}, \; \mathrm{supp}(\mathcal{E}_j) \cap U_{V,\ell} \neq \emptyset \right\}.$$

Then

$$\mathcal{H}^{\langle \ell \rangle} := \big(V, \mu_V, \nu_V, \mathfrak{E}^{\langle \ell \rangle}\big)$$

is a RIF-(n,ℓ)-SHG on V_0. We call $\mathcal{H}^{\langle \ell \rangle}$ the level-induced *(or* depth-ℓ truncated*) structure of $\mathcal{H}$.*

In particular,

$$\mathcal{H}^{\langle 0 \rangle}$$

is an intuitionistic fuzzy n-SuperHyperGraph (non-recursive edge level), and

$$\mathcal{H}^{\langle k \rangle} = \mathcal{H}.$$

Proof. Because the supervertex data (V, μ_V, ν_V) are unchanged, conditions Definition 3.3(i)–(ii) remain valid. Each truncated edge $\mathcal{E}_j^{\langle \ell \rangle}$ is an intuitionistic fuzzy set on $U_{V,\ell}$, and Atanassov's constraint is inherited by restriction.

The vertex–edge consistency condition is also inherited, since for any $v \in V = S_0 \subseteq U_{V,\ell}$,

$$\mu_j|_{U_{V,\ell}}(v) = \mu_j(v) \le \mu_V(v), \qquad \nu_j|_{U_{V,\ell}}(v) = \nu_j(v) \ge \nu_V(v).$$

The covering condition on V is unchanged because all supervertices lie in $S_0 \subseteq U_{V,\ell}$, so the values on V are identical before and after truncation.

Therefore, $\mathcal{H}^{\langle \ell \rangle}$ satisfies all axioms of a RIF-(n,ℓ)-SHG. For $\ell = 0$, one has $U_{V,0} = V$, so recursion disappears and the structure reduces to an intuitionistic fuzzy n-SuperHyperGraph. For $\ell = k$, the truncation is the identity. □

Conclusion

In this paper, we introduced *Recursive Intuitionistic Fuzzy SuperHyperGraphs* as a unified framework that combines hierarchical supervertex structures, recursive superhyperedge formation, and intuitionistic fuzzy uncertainty. For future work, we expect that this framework can be extended in several directions, including variants based on *Neutrosophic Graphs* [21,22] and *Plithogenic Graphs* [23]. We also anticipate quantitative investigations supported by computational experiments, which may help clarify the structural behavior and practical applicability of these models in complex uncertain systems.

Competing interests

All the authors declare that there are no conflicts of interest.

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